

EULER-MAHONIAN STATISTICS AND DESCENT BASES FOR SEMIGROUP ALGEBRAS

BENJAMIN BRAUN AND MCCABE OLSEN

ABSTRACT. We consider quotients of the unit cube semigroup algebra by particular $\mathbb{Z}_r \wr S_n$ -invariant ideals. Using Gröbner basis methods, we show that the resulting graded quotient algebra has a basis where each element is indexed by colored permutations $(\pi, \epsilon) \in \mathbb{Z}_r \wr S_n$ and each element encodes the negative descent and negative major index statistics on (π, ϵ) . This gives an algebraic interpretation of these statistics which was previously unknown. This basis of the $\mathbb{Z}_r \wr S_n$ -quotients allows us to recover certain combinatorial identities involving Euler-Mahonian distributions of statistics.

1. INTRODUCTION

Let $[0, 1]^n \subset \mathbb{R}^n$ denote the the n -dimensional unit cube. Let S_n denote the symmetric group on n elements. Let $[n] := \{1, 2, \dots, n\}$.

1.1. Polytope semigroup algebras. Let $\mathcal{P} \subset \mathbb{R}^n$ be an n -dimensional convex lattice polytope and suppose that $\mathcal{P} \cap \mathbb{Z}^n = \{p_1, p_2, \dots, p_k\}$. For this paper, assume that P satisfies the integer decomposition property, i.e. that for every positive integer k , every lattice point in the k -th dilate of P is a sum of exactly k lattice points in P . The *affine semigroup algebra of \mathcal{P}* over \mathbb{C} is

$$\mathbb{C}[\mathcal{P}] := \mathbb{C}[t \cdot \mathbf{x}^{p_i} : 1 \leq i \leq k] \subset \mathbb{C}[t, x_1^{\pm 1}, x_2^{\pm 1}, \dots, x_n^{\pm 1}],$$

where $\mathbf{x}^{p_i} = x_1^{p_{i1}} x_2^{p_{i2}} \dots x_n^{p_{in}}$ when $p_i = (p_{i1}, p_{i2}, \dots, p_{in})$. This semigroup algebra is generated by the lattice points in the *cone over \mathcal{P}* , which is

$$\text{cone}(\mathcal{P}) := \text{span}_{\mathbb{R}_{\geq 0}}\{(1, p) : p \in \mathcal{P}\}.$$

For greater detail and background of semigroup algebras and cones over polytopes, see [16].

Let $\mathcal{P} = [0, 1]^n$, which is known to satisfy the integer decomposition property. Let $R_n := \mathbb{C}[[0, 1]^n]$ denote the affine semigroup algebra of $[0, 1]^n$ which has the following description:

$$R_n = \mathbb{C}[t \cdot x_{a_1} \dots x_{a_i} \mid A = \{a_1, \dots, a_i\} \subseteq [n]] \subset \mathbb{C}[t, x_1, x_2, \dots, x_n].$$

Alternatively, we can define we can define R_n as the quotient of a polynomial ring by a toric ideal. Let T_n be a polynomial ring in 2^n variables, where each variable corresponds to a subset of $[n]$, thus

$$T_n := \mathbb{C}[z_A : A \subseteq [n]].$$

Define the toric ideal

$$I_n := \langle z_A z_B - z_{A \cap B} z_{A \cup B} \mid A \not\subseteq B \text{ and } B \not\subseteq A \rangle.$$

It is known that $R_n \cong T_n/I_n$. For background and details see [16]. This algebra also arises as the Hibi ring for the antichain on n elements, as the unit cube is the order polytope of the antichain (see e.g. [8, 12, 13] for additional details of Hibi rings). We will use R_n to denote T_n/I_n when it

Date: 22 September 2016.

2010 *Mathematics Subject Classification.* Primary: 52B20, 13P10, 05A19, 05E40 Secondary: 05A05, 05E05.

The first author was partially supported by grant H98230-16-1-0045 from the U.S. National Security Agency. The second author was partially supported by a 2016 National Science Foundation/Japanese Society for the Promotion of Science East Asia and Pacific Summer Institutes Fellowship award NSF OISE-1613525.

is convenient. It is well-known that the Hilbert series of R_n with respect to grading by t -degree is $\sum_{k \geq 0} (k+1)^n t^k$, leading us to the topic of Euler-Mahonian identities.

1.2. Euler-Mahonian identities. We first review certain permutation statistics. For $\pi \in S_n$ with $\pi = \pi_1 \pi_2 \dots \pi_n$, the *descent set* is defined to be

$$\text{Des}(\pi) := \{i \in \{1, 2, \dots, n-1\} : \pi_i > \pi_{i+1}\}.$$

Moreover, the *descent number* is $\text{des}(\pi) := \#\text{Des}(\pi)$. The descent number is encoded in the *Eulerian polynomial* $A_n(t) := \sum_{\pi \in S_n} t^{\text{des}(\pi)}$ which satisfies the identity

$$(1) \quad \sum_{k \geq 0} (k+1)^n t^k = \frac{A_n(t)}{(1-t)^{n+1}},$$

first studied by Euler [9]. This identity was generalized to a bivariate identity usually attributed to Carlitz using the major index; see [5] and the references therein for more details on the history of these identities. Given $\pi \in S_n$, the *major index* of π is defined to be

$$\text{maj}(\pi) := \sum_{j \in \text{Des}(\pi)} j.$$

Theorem 1.1 (Carlitz, [6]). *For all $n \geq 1$,*

$$\sum_{k \geq 0} [k+1]_q^n t^k = \frac{\sum_{\pi \in S_n} t^{\text{des}(\pi)} q^{\text{maj}(\pi)}}{\prod_{j=0}^n (1 - tq^j)}$$

where $[k+1]_q = 1 + q + q^2 + \dots + q^k$.

This identity, which we will call the *Euler-Mahonian identity*, has arisen in a variety of contexts since its inception. Some such scenarios include lecture hall partition generating function identities [17], polyhedral-geometric studies of the semigroup algebra for $\text{cone}([0, 1]^n)$ [5], Hilbert series related to a descent basis for the coinvariant algebra of S_n [2], 0-Hecke algebra actions on Stanley-Reisner rings [15], and quasisymmetric function identities [18].

Generalizing to colored permutations groups $\mathbb{Z}_r \wr S_n$, one can consider the *flag statistics* as well as the *negative statistics*, the latter of which we define in Section 2. These statistics were originally introduced for the hyperoctohedral group $B_n \cong \mathbb{Z}_2 \wr S_n$ [1] and generalized for $r \geq 2$ to $\mathbb{Z}_r \wr S_n$ [3, 4]; for these families of statistics, the following Euler-Mahonian identities exist.

Theorem 1.2 (Bagno, [3]). *Given any $r \geq 2$, for all $n \geq 1$,*

$$\sum_{k \geq 0} [k+1]_q^n t^k = \frac{\sum_{(\rho, \epsilon) \in \mathbb{Z}_r \wr S_n} t^{\text{ndes}(\rho, \epsilon)} q^{\text{nmajor}(\rho, \epsilon)}}{(1-t) \prod_{j=1}^n (1 - t^r q^{rj})}$$

Theorem 1.3 (Bagno-Biagioli, [4]). *Given any $r \geq 2$, for all $n \geq 1$,*

$$\sum_{k \geq 0} [k+1]_q^n t^k = \frac{\sum_{(\rho, \epsilon) \in \mathbb{Z}_r \wr S_n} t^{\text{fdes}(\rho, \epsilon)} q^{\text{fmajor}(\rho, \epsilon)}}{(1-t) \prod_{j=1}^n (1 - t^r q^{rj})}$$

1.3. Our Contributions. The goal of this paper is twofold. First, we produce a new algebraic interpretation of negative permutation statistics by considering $\mathbb{Z}_r \wr S_n$ -quotient algebras of R_n . To do so, we consider an ideal $\text{invar}(r, n) \subset R_n$ which is generated by certain invariants of R_n under a $\mathbb{Z}_r \wr S_n$ -action, defined in detail in Section 3. We obtain the following theorem using Gröbner basis techniques.

Theorem 1.4 (see Theorem 4.1). *There exists a basis of $R_n/\overline{\text{invar}(r, n)}$ of the form $\{b_{(\sigma, X)}^r + \overline{\text{invar}(r, n)}\}$ with elements indexed by pairs (σ, X) that are in bijection with colored permutations $(\pi, \epsilon) \in \mathbb{Z}_r \wr S_n$. Further, $b_{(\sigma, X)}^r$ encodes $\text{ndes}(\pi, \epsilon)$ and $\text{nmajor}(\pi, \epsilon)$. The bijective correspondence of $(\sigma, X) \leftrightarrow (\pi, \epsilon)$ is given in Remark 2.5.*

Our second goal is to consider a multigraded Hilbert series of R_n and the quotient $R_n/\overline{\text{invar}(r, n)}$. These computations allow us to recover the identities given by Theorem 1.1 and Theorem 1.2. These new proofs provide a new perspective on identities of this type.

Moreover, the new proof of Theorem 1.1 serves to forge connections between the commutative-algebraic and representation-theoretic methods [2] for the S_n -coinvariant algebra $\mathbb{C}[x_1, \dots, x_n]/\mathcal{I}_n$, where $\mathcal{I}_n := \langle e_1, \dots, e_n \rangle$ with e_i denoting the i -th elementary symmetric function, and polyhedral-geometric methods for $\text{cone}([0, 1]^n)$ [5]. Additionally, we provide a short proof that this quotient algebra is isomorphic as a graded S_n -module to the S_n -coinvariant algebra $\mathbb{C}[x_1, \dots, x_n]/\mathcal{I}_n$. We believe that these results, like those given in [5], support the idea that $\text{cone}([0, 1]^n)$ and its associated semigroup algebra are analogues of the polynomial ring in n variables that give rise to interesting and different structures and results in similar contexts.

2. COLORED PERMUTATION GROUPS AND DECENT SETS

The wreath product $\mathbb{Z}_r \wr S_n$ of a cyclic group of order r with S_n consists of pairs (π, ϵ) where $\pi \in S_n$ and $\epsilon \in \{\omega^0, \omega^1, \dots, \omega^{r-1}\}^n$ for $\omega := e^{2\pi i/r}$ a primitive r th root of unity. These groups are often called *colored permutation groups* and the elements are commonly referred to as *colored* or *indexed* permutations. We adopt the usual window notation, denoting the pair (π, ϵ) by $[\pi(1)^{c_1} \pi(2)^{c_2} \dots \pi(n)^{c_n}]$ where $\epsilon_j = \omega^{c_j}$. Additionally, we will use the notation j^{c_j} and (ω^{c_j}, j) to denote elements of $\{\omega^0, \omega^1, \dots, \omega^{r-1}\} \times [n]$.

Elements $(\pi, \epsilon) \in \mathbb{Z}_r \wr S_n$ can be identified as a permutation matrix for π where the 1 in position $(\pi(i), i)$ is replaced with ϵ_i . The algebraic structure of $\mathbb{Z}_r \wr S_n$ is described by matrix multiplication where entry-by-entry multiplication of the nonzero entries is given by the group operation of \mathbb{Z}_r . A more explicit understanding of these wreath products may be found in [3, 4, 5, 17].

To review one definition of descents for wreath products, we define a total order as follows. Given $j^{c_j}, k^{c_k} \in \{\omega^0, \omega^1, \dots, \omega^{r-1}\} \times [n]$ we say that $j^{c_j} < k^{c_k}$ if $c_j > c_k$ or if $c_j = c_k$ and $j < k$ hold.

Definition 2.1. Let $(\pi, \epsilon) \in \mathbb{Z}_r \wr S_n$. The *type-A descent set* is defined to be

$$\text{Des}_A(\pi, \epsilon) := \{i \in [n-1] : \pi_i^{c_i} > \pi_{i+1}^{c_{i+1}}\}$$

and the *type-A descent statistic* is

$$\text{des}_A(\pi, \epsilon) := \#\text{Des}_A(\pi, \epsilon).$$

The *type-A major index* is

$$\text{major}_A(\pi, \epsilon) := \sum_{j \in \text{Des}_A(\pi, \epsilon)} j$$

We now review the *negative* statistics on $\mathbb{Z}_r \wr S_n$.

Definition 2.2. For an element $(\pi, \epsilon) \in \mathbb{Z}_r \wr S_n$, we define the *negative inverse multiset* as

$$\text{NNeg}(\pi, \epsilon) := \underbrace{\{i, i, \dots, i : i \in [n]\}}_{c_i \text{ times}}.$$

The *negative descent multiset* is

$$\text{NDes}(\pi, \epsilon) := \text{Des}_A(\pi, \epsilon) \cup \text{NNeg}((\pi, \epsilon)^{-1}).$$

The *negative descent statistic* is

$$\text{ndes}(\pi, \epsilon) := \#\text{NDes}(\pi, \epsilon).$$

The *negative major index* is

$$\text{nmajor}(\pi, \epsilon) := \sum_{i \in \text{NDes}(\pi, \epsilon)} i.$$

We will use the following representation for elements of $\mathbb{Z}_r \wr S_n$.

Definition 2.3. The *increasing elements* of $\mathbb{Z}_r \wr S_n$, denoted $\mathcal{I}_{r,n}$, is the subset of elements satisfying $\text{des}_A(\pi, \epsilon) = 0$.

It is a simple exercise to see that any element of $\mathbb{Z}_r \wr S_n$ can be represented uniquely as

$$(\pi, \epsilon) \circ (\sigma, (1, 1, \dots, 1))$$

for some $\sigma \in S_n$ and $(\pi, \epsilon) \in \mathbb{Z}_r \wr S_n$. Subsequently, we have that

$$\mathbb{Z}_r \wr S_n = \bigcup_{\sigma \in S_n} \mathcal{I}_{r,n} \sigma$$

where we use σ in place of $(\sigma, (1, 1, \dots, 1))$ for ease.

We also have the following observation.

Proposition 2.4. [5, Proposition 5.11] For $(\rho, \delta) \in \mathcal{I}_{r,n}$ and $\sigma \in S_n$,

$$\text{NNeg}([\rho, \delta] \sigma^{-1}) = \text{NNeg}((\rho, \delta)^{-1}).$$

Further, each permutation $(\rho, \delta) \in \mathcal{I}_{r,n}$ is uniquely determined by $\text{NNeg}((\rho, \delta)^{-1})$.

Remark 2.5. We will often denote $(\pi, \epsilon) \in \mathbb{Z}_r \wr S_n$ by the pair (σ, X) where $\sigma \in S_n$ satisfies $(\rho, \delta)\sigma = (\pi, \epsilon)$ with $(\rho, \delta) \in \mathcal{I}_{r,n}$ and $X = \text{NNeg}((\pi, \epsilon)^{-1})$. This establishes a bijective correspondence between elements of $\mathbb{Z}_r \wr S_n$ and pairs (σ, X) with $\sigma \in S_n$ and X a multiset of elements of $[n]$ in which each element appears with multiplicity strictly less than r . For convenience of notation, we will write $(\sigma, X) \in \mathbb{Z}_r \wr S_n$ when this interpretation is preferred.

3. $\mathbb{Z}_r \wr S_n$ -QUOTIENT ALGEBRAS OF R_n AND DESCENT BASES

For convenience, we will view $R_n \cong T_n/I_n$ as the quotient of a polynomial ring by the toric ideal I_n . First consider the S_n case. We define an S_n action on T_n given as $S_n \times T_n \rightarrow T_n$ defined on the variables by $(\pi, z_A) \mapsto z_{\pi(A)} = z_{\{\pi(a_1), \dots, \pi(a_k)\}}$ where $A = \{a_1, \dots, a_k\}$. Note that this action passes to $R_n \cong T_n/I_n$, where it corresponds to the usual action of permutations on $\{x_1, \dots, x_n\}$. We consider the following ideal of elements which are invariant under this action:

$$\text{invar}(1, n) := \left\langle \hat{e}_k := \sum_{|A|=k} z_A \mid \text{for all } 0 \leq k \leq n \right\rangle$$

The elements \hat{e}_k are the T_n -analogue to the usual elementary symmetric functions e_k in the polynomial ring on n variables. It is important to note that this ideal is not the full ideal of invariants in T_n , as there must be 2^n algebraically independent invariants [20, Proposition 2.1.1]. However, the generators are indeed invariant and this is the appropriate ideal for our purposes. We say the S_n quotient algebra of R_n is $R_n/\overline{\text{invar}(1, n)}$, where $\overline{\text{invar}(1, n)}$ is the image of $\text{invar}(1, n)$ in the quotient T_n/I_n . For convenience, we will consider the ring $T_n/J_{1,n}$ where $J_{1,n} := \text{invar}(1, n) + I_n$, as it is a straightforward exercise in algebra to show that $T_n/J_{1,n} \cong R_n/\overline{\text{invar}(1, n)}$.

Next, we consider $\mathbb{Z}_r \wr S_n$ for $r \geq 2$. Consider the action $\mathbb{Z}_r \wr S_n \times T_n \rightarrow T_n$ defined on the variables by $((\pi, \epsilon), z_A) \mapsto (\prod_{i \in A} \epsilon_i) \cdot z_{\pi(A)} = (\prod_{i \in A} \epsilon_i) \cdot z_{\{\pi(a_1), \dots, \pi(a_k)\}}$ where $A = \{a_1, \dots, a_k\}$. We consider an ideal generated by invariant elements of this action:

$$\text{invar}(r, n) := \left\langle z_\emptyset, \hat{e}_k^r := \sum_{|A|=k} z_A^r \mid \text{for all } 1 \leq k \leq n \right\rangle$$

This is consistent with the above in the $r = 1$ case. This ideal also does not contain all of the invariants of T_n under this action, but the ideal is the appropriate choice of invariant generators for our scenario. We say the $\mathbb{Z}_r \wr S_n$ quotient algebra of R_n is $R_n/\overline{\text{invar}(r, n)}$, and we will consider the ring $T_n/J_{r,n}$ where $J_{r,n} := \text{invar}(r, n) + I_n$, as we have $T_n/J_{r,n} \cong R_n/\overline{\text{invar}(r, n)}$.

Now, we will define descent bases for our quotients. First consider $T_n/J_{1,n}$. We wish to construct a basis based on the descent sets of S_n that is analogous to the Garsia-Stanton descent basis. The Garsia-Stanton descent basis is a basis for the S_n -coinvariant algebra $\mathbb{C}[x_1, \dots, x_n]/\mathcal{I}_n$ with coset representatives

$$a_\pi = \prod_{j \in \text{Des}(\pi)} x_{\pi(1)} \cdots x_{\pi(j)}$$

for all $\pi \in S_n$. Garsia and Stanton originally showed this was a basis in [10] using the theory of Stanley-Reisner rings. In [2], Adin, Brenti, and Roichman provide another proof of this result and use the basis heavily in their proof of the Euler-Mahonian identity for S_n . We introduce an analogue of the Garsia-Stanton basis for $T_n/J_{1,n}$, which is

$$\hat{a}_\pi := \prod_{j \in \text{Des}(\pi)} z_{\{\pi(1), \pi(2), \dots, \pi(j)\}}$$

for all $\pi \in S_n$. Because of the correspondence given in Theorem 6.1, in this paper we will refer to the set $\{\hat{a}_\pi : \pi \in S_n\}$ as the *Garsia-Stanton basis*. Using Gröbner basis arguments in Section 4, we will show that this is indeed a basis for $T_n/J_{1,n}$.

We can generalize this to a basis of $T_n/J_{r,n}$ for $r \geq 2$.

Definition 3.1. The *negative descent basis* of $T_n/J_{r,n}$ consists of the elements

$$b_{(\sigma, X)}^r := \hat{a}_\sigma \cdot \prod_{j \in X} z_{\{\sigma(1), \sigma(2), \dots, \sigma(j)\}}$$

for all $\sigma \in S_n$ and X a multiset of $[n]$ where no element has multiplicity greater than $r - 1$.

We will show that this is a basis in Section 4. It follows from Remark 2.5 that if (σ, X) corresponds to $(\rho, \epsilon) \in \mathbb{Z}_r \wr S_n$, then $\text{NNeg}((\rho, \epsilon)^{-1}) = X$ and $\text{Des}_A(\rho, \epsilon) = \text{Des}(\sigma)$. So, elements of this basis correspond to NDes sets of $\mathbb{Z}_r \wr S_n$, hence the name “negative descent basis.” It is important to observe that this is distinct from the basis developed by R. Adin, F. Brenti, and Y. Roichman [2] for the hyperoctohedral group $B_n \cong \mathbb{Z}_2 \wr S_n$, as their basis related to the *flag descent sets*.

4. DESCENT BASES VIA GRÖBNER BASES FOR $J_{r,n}$

Our goal in this section is to prove the following theorem.

Theorem 4.1. For $r \geq 2$, $\{b_{(\sigma, X)}^r : (\sigma, X) \in \mathbb{Z}_r \wr S_n\}$ is a basis of $T_n/J_{r,n}$. When $r = 1$, $\{\hat{a}_\pi : \pi \in S_n\}$ is a basis of $T_n/J_{1,n}$.

Theorem 4.1 is an immediate consequence of Proposition 4.2 and Theorem 4.4.

Proposition 4.2. Suppose that the negative descent basis contains precisely those monomials outside of a leading term ideal for $J_{r,n}$ for some term order (for $r = 1$, this is the Garsia-Stanton basis described above). Then the leading terms for that term order must be the following:

- z_\emptyset
- z_A^r , where $A = [k]$ for all $1 \leq k \leq n$
- z_A^{r+1} where $A \neq [k]$ for any $0 \leq k \leq n$
- $z_A z_B$ for any $0 \leq k \leq n$, such that $A \not\subseteq B$ and $B \not\subseteq A$
- $z_A^r z_B$ where $A \neq [k]$ for any $0 \leq k \leq n$, such that $A \subset B$ and $\min(B \setminus A) > \max(A)$
- $z_A z_B^r$ where $B \neq [k]$ for any $0 \leq k \leq n$, such that $A \subset B$ and there is an ℓ with $[\ell] \not\subseteq A$, $[\ell] \subset B$, and $B \setminus A \subset [\ell]$

- $z_{A_1} z_{A_2}^r z_{A_3}$, where $A_2 \neq [k]$ for any $0 \leq k \leq n$, such that $A_1 \subset A_2 \subset A_3$ and $\max(A_2 \setminus A_1) < \min(A_3 \setminus A_2)$

Proof. We will first show the argument for $r = 1$, the Garsia-Stanton basis, then we will generalize the argument for $r \geq 2$. Assume unless otherwise stated that elements of sets are written in ascending order, e.g. $A = \{a_1, a_2, \dots, a_\ell\}$ implies $a_1 < a_2 < \dots < a_\ell$. First, note that the following observations imply that every monomial \hat{a}_π is not divisible by any of the claimed leading term monomials.

- z_\emptyset clearly cannot divide \hat{a}_π by construction.
- $z_{\{1,2,3,\dots,k\}}$ cannot divide \hat{a}_π , as this would imply that there is a descent at the position k , but there is no element smaller than k which has not already appeared.
- z_A^2 cannot divide \hat{a}_π , as by definition each set A which arises from $\text{Des}(\pi)$ must be unique.
- By definition, if $z_A z_B$ is a factor of \hat{a}_π , it implies that $A \subset B$ or vice versa.
- If $z_A z_B$ divides \hat{a}_π with $A \subset B$ such that $A = \{a_1, a_2, \dots, a_\ell\}$ and $B = A \cup \{b_1, \dots, b_k\}$, we must have that $b_1 < a_\ell$ else there is no descent possible at position ℓ .
- If $z_A z_B$ divides \hat{a}_π with $A \subset B$ such that $[\ell] \not\subset A$ and $[\ell] \subset B$, we must have some element $x \in B \setminus A$ such that $x \notin [\ell]$, else no descent could occur since $\pi|_{[B]} \in [\ell]$ and $[\ell] \subset \{\pi(1), \dots, \pi(|B|)\}$.
- If $z_{A_1} z_{A_2} z_{A_3}$ divides \hat{a}_π where $A_1 \subset A_2 \subset A_3$ and $\max(A_2 \setminus A_1) < \min(A_3 \setminus A_2)$, then no descent could occur between set A_2 and A_3 , i.e. in position $\pi(|A_2|)$.

Suppose next that we have a monomial in $m \in T_n$, which is divisible by none of our claimed leading term monomials. We claim that there exists some $\pi \in S_n$ such that $m = \hat{a}_\pi$; to prove this claim, first we write

$$m = z_{B_1} z_{B_2} \cdots z_{B_s}$$

where $B_1 \subset B_2 \subset \dots \subset B_s$. We denote $B_1 = \{\beta_{1_1}, \dots, \beta_{1_{m_1}}\}$ and $B_i = B_{i-1} \cup \{\beta_{i_1}, \dots, \beta_{i_{m_i}}\}$ for all $1 < i \leq s$. Note that this union corresponds to the permutation

$$\pi = \beta_{1_1} \cdots \beta_{1_{m_1}} \beta_{2_1} \cdots \beta_{2_{m_2}} \cdots \beta_{s_1} \cdots \beta_{s_{m_s}} \gamma_1 \cdots \gamma_t$$

where $\gamma_1 < \gamma_2 < \dots < \gamma_t$ are the elements which do not appear in any B_i set. Moreover, we have that $\beta_{i_{m_i}} > \beta_{i+1_1}$ and $\beta_{s_{m_s}} > \gamma_1$ and these will be the only such descents since m is not divisible by any of the claimed leading term monomials. Hence, m is a Garsia-Stanton descent element \hat{a}_π . (This argument is similar to standard P -partition arguments [19, Lemma 3.15.3].)

Now suppose that $r \geq 2$. By a similar argument to that just given, $b_{(\pi, X)}^r$ is not divisible by a monomial from among the claimed leading terms, since:

- z_\emptyset clearly cannot divide $b_{(\pi, X)}^r$ by construction.
- $z_{[k]}^r$ cannot appear in $b_{(\pi, X)}^r$, as, since the greatest possible multiplicity of any element in X is $r - 1$, this would imply that there is a position k descent in π when all smaller elements than $\pi(k)$ have already appeared in π .
- z_A^{r+1} for $A \neq [k]$ cannot appear in $b_{(\pi, X)}^r$ as we only obtain a single z_A from \hat{a}_π , and we can obtain at most $r - 1$ copies of z_A from the product over X . Note that if z_A^r appears in $b_{(\pi, X)}^r$, then one of the z_A terms must have come from the product indexed by $\text{Des}(\pi)$, and thus $|A| \in \text{Des}(\pi)$.
- By definition, $z_A z_B$ a factor of $b_{(\pi, X)}^r$ implies $A \subseteq B$ or vice-versa.
- If $z_A^r z_B$ appears in $b_{(\pi, X)}^r$ where $A \subset B$ with $A = \{a_1, a_2, \dots, a_\ell\}$ and $B = A \cup \{b_1, \dots, b_k\}$, it follows that $|A| \in \text{Des}(\pi)$, thus we must have that $b_1 < a_\ell$ else there is no descent occurring in π in position $|A|$.
- If $z_A z_B^r$ appears in $b_{(\pi, X)}^r$ where $A \subset B$ with $[\ell] \not\subset A$ and $[\ell] \subset B$, then $|B| \in \text{Des}(\pi)$. Hence, there must exist an element $x \in B \setminus A$ such that $x \notin [\ell]$, else no descent can occur.

- If $z_{A_1} z_{A_2}^r z_{A_3}$ appears in $b_{(\pi, X)}^r$ such that $A_1 \subset A_2 \subset A_3$ and $\max(A_2 \setminus A_1) < \min(A_3 \setminus A_2)$, then it must be that $|A_2| \in \text{Des}(\pi)$ and hence no descent can occur between set A_2 and A_3 , i.e. in position $\pi(|A_2|)$.

Suppose next that we have a monomial in $m_r \in T_n$, which is divisible by none of our claimed leading term monomials. We claim that there exists some $\pi \in S_n$ and X a multiset of $[n]$ with every element having multiplicity strictly less than r such that $m_r = b_{(\pi, X)}^r$; to prove this claim, first we write

$$m_r = z_{B_1}^{b_1} z_{B_2}^{b_2} \cdots z_{B_s}^{b_s}$$

where we have $B_1 \subset B_2 \subset \cdots \subset B_s$. Note that $b_i \leq r$ if $B_i \neq [k]$ and $b_i \leq r - 1$ if $B_i = [k]$. As in the previous case, inductively define $B_i = B_{i-1} \cup \{\beta_{i_1}, \dots, \beta_{i_{m_i}}\}$. Construct a new monomial

$$m'_r = z_{B_1} z_{B_2} \cdots z_{B_s}$$

and the set

$$\tilde{X} = \{\underbrace{c_1, \dots, c_1}_{b_1-1 \text{ times}}, \underbrace{c_2, \dots, c_2}_{b_2-1 \text{ times}}, \dots, \underbrace{c_s, \dots, c_s}_{b_s-1 \text{ times}}\}$$

where $c_i = |B_i|$. We associate to m'_r the permutation

$$\pi = \beta_{1_1} \cdots \beta_{1_{m_1}} \beta_{2_1} \cdots \beta_{2_{m_2}} \cdots \beta_{s_1} \cdots \beta_{s_{m_s}} \gamma_1 \cdots \gamma_t$$

where $\gamma_1 < \gamma_2 < \cdots < \gamma_t$ are the elements which do not appear in any B_i set. Since the β -values within each B_i are increasing, only possible descents occur between $\beta_{i_{m_i}}$ and β_{i+1_1} . If we have $\beta_{i_{m_i}} > \beta_{i+1_1}$, then we have a descent and we do nothing. (Note that the final three claimed leading term monomials force a descent to occur if b_i takes on a maximal value of r or $r - 1$, showing that all seven of the claimed leading term monomial conditions are required for this argument to hold.) If we have $\beta_{i_{m_i}} < \beta_{i+1_1}$, then there is no descent and we consequently adjust m'_r to be

$$m'_{r, \setminus i} = z_{B_1} \cdots z_{B_{i-1}} z_{B_i+1} \cdots z_{B_s}.$$

If we iteratively do this for all such situations where no descent occurs, and call the resulting monomial \widetilde{m}_r , we have that $\hat{a}_\pi = \widetilde{m}_r$ by our argument in the $r = 1$ case. Moreover, we set $X := \tilde{X} \cup \{c_i : \text{where } \beta_{i_{m_i}} < \beta_{i+1_1}\}$, where as before $c_i = |B_i|$. With this choice of permutation and multiset we obtain $m = b_{(\pi, X)}^r$. \square

Definition 4.3. Given two sets A and B such that $|A| = |B| = k$, we say that A is *lexicographically before* B if there exists $i \in A$ s.t. $i \notin B$ and given any $j \in B$ such that $j < i$ we have $j \in A$.

For example, the ordering of 3-subsets of the 5-set would be $1, 2, 3 < 1, 2, 4 < 1, 2, 5 < 1, 3, 4 < 1, 3, 5 < 1, 4, 5 < 2, 3, 4 < 2, 3, 5 < 2, 4, 5 < 3, 4, 5$. Our next step is to prove that the leading terms listed in Proposition 4.2 arise when the variables of T_n are given the linear order $z_A > z_B$ if $|A| < |B|$ or if $|A| = |B|$ and A is lexicographically before B , and the graded reverse lexicographic (or *grevlex*) monomial ordering is imposed. We use $\text{LT}(I)$ to denote the leading term ideal of an ideal I . For background regarding Gröbner bases see [7]. For convenience, throughout the following proof we will refer to a pair of subsets A and B such that $A \not\subseteq B$ and $B \not\subseteq A$ as a *Sperner 2-pair*.

Theorem 4.4. *There exists a Gröbner basis $G_{r,n}$ of $J_{r,n}$ for which $\text{LT}(G_{r,n})$ is the ideal generated by terms of the form listed in Proposition 4.2.*

Proof. Use the term order for T_n described above. Our proof will involve computing S -polynomials starting from the generators of $J_{r,n}$. To minimize the number of computations required, we first make a dimension argument showing that the number of monomials outside of the leading term ideal for $J_{r,n}$ is the number of elements of the negative descent basis. We then compute S -polynomials to produce elements with all of the leading terms listed in Proposition 4.2, which will complete

the proof. We will compute the S -polynomials for arbitrary r , but we will make two dimension arguments, for $r = 1$ and $r \geq 2$.

Consider $r = 1$. Combining an elementary exercise and [19, Proposition 1.4.4] we see that the Hilbert series for R_n is given by

$$\text{Hilb}(R_n; t) = \sum_{k \geq 0} (k+1)^n t^n = \frac{A_n(t)}{(1-t)^{n+1}}$$

where $A(n) = \sum_{\pi \in S_n} t^{\text{des}(\pi)}$ is the Eulerian polynomial. Let $\mathcal{C}_{1,n} := \mathbb{C}[\hat{e}_k + I_n | 0 \leq k \leq n]$, and note that the elements $\hat{e}_k + I_n$ are algebraically independent since they specialize in R_n (by setting $t = 1$) to the usual elementary symmetric functions; note that $\text{Hilb}(\mathcal{C}_{1,n}; t) = \frac{1}{(1-t)^{n+1}}$. Hochster's Theorem implies that R_n is Cohen-Macaulay [14], and since $\text{invar}(1, n)$ is an ideal generated by an algebraically independent system of parameters, we have

$$\text{Hilb}(T_n/J_{1,n}; t) = A_n(t)$$

by [12, Lemma 17.1]. The \mathbb{C} -dimension of $T_n/J_{1,n}$ is

$$\dim_{\mathbb{C}}(T_n/J_{1,n}) = \text{Hilb}(T_n/J_{1,n}; 1) = A(1) = n!,$$

which is the number of elements in the Garsia-Stanton descent basis, as desired.

Now, suppose that $r \geq 2$. Let $\mathcal{C}_{r,n} = \mathbb{C}[z_{\emptyset} + I_n, \hat{e}_k^r + I_n | 1 \leq k \leq n]$. Given that R_n is Cohen-Macaulay and that $\hat{e}_k^r + I_n$ and $z_{\emptyset} + I_n$ are algebraically independent, hence $\text{Hilb}(\mathcal{C}_{r,n}; t) = \frac{1}{(1-t)(1-t^r)^n}$, we have that

$$\text{Hilb}(R_n; t) = \sum_{k \geq 0} (k+1)^n t^n = \frac{B_{r,n}(t)}{(1-t)(1-t^r)^n}$$

where $B_{r,n}(t) = A_n(t) \cdot (1+t+\cdots+t^{r-1})^n$ by our previous calculation for $r = 1$. Thus,

$$\text{Hilb}(T_n/J_{r,n}; t) = A_n(t) \cdot (1+t+\cdots+t^{r-1})^n$$

from which we can conclude that

$$\dim_{\mathbb{C}}(T_n/J_{r,n}) = \text{Hilb}(T_n/J_{r,n}; 1) = B_{r,n}(1) = r^n n!,$$

which is the number of elements in the negative descent basis, as desired.

Next, we move to S -polynomial calculations. Our goal is to compute S -polynomials until all the elements listed in Proposition 4.2 arise as leading terms; since at that point we will have reached the correct value of $\dim_{\mathbb{C}}(T_n/J_{r,n}) = \dim_{\mathbb{C}}(T_n/\text{LT}(J_{r,n}))$, we must have a Gröbner basis.

We begin by noting that some of our desired leading terms arise from the generators of $J_{r,n}$. First, $z_A z_B$ such that $A \not\subset B$ and $B \not\subset A$ where $A \neq [k] \neq B$ for any k are leading terms of I_n . The monomials z_{\emptyset} and z_A^r where $A = [k]$ for $k = 1, \dots, n$ are the leading terms of $\text{invar}(r, n)$. These account for the fourth, first, and second items listed in Proposition 4.2, respectively.

To obtain an element with the leading term z_A^{r+1} as given in the third bullet of Proposition 4.2, suppose that $|A| = k$ and consider the following S -polynomial:

$$\begin{aligned} & S(\hat{e}_k^r, z_{[k]} z_A - z_{[k] \cap A} z_{[k] \cup A}) \\ &= \frac{z_{[k]}^r z_A}{z_{[k]}^r} \left(z_{[k]}^r + z_{A_1}^r + z_{A_2}^r + \cdots + z_A^r + \cdots + z_{A_{\binom{n}{k}-1}}^r \right) \\ & \quad - \frac{z_{[k]}^r z_A}{z_{[k]} z_A} (z_{[k]} z_A - z_{[k] \cap A} z_{[k] \cup A}) \\ &= z_A \left(z_{A_1}^r + z_{A_2}^r + \cdots + z_A^r + \cdots + z_{A_{\binom{n}{k}-1}}^r \right) + z_{[k]}^{r-1} z_{[k] \cap A} z_{[k] \cup A} \end{aligned}$$

Note that the term order implies that

$$z_A z_{A_1}^r > z_A z_{A_2}^r > \cdots > z_A z_A^r > \cdots z_A z_{A_{\binom{n}{k}-1}}^r > z_{[k]}^{r-1} z_{[k] \cap A} z_{[k] \cup A}$$

However, for each i where $A_i \neq A$, $z_A z_{A_i}$ is the leading term of a polynomial of $J_{r,n}$, and we use $z_A z_{A_i} - z_{A \cap A_i} z_{A \cup A_i} \in J_{r,n}$ to rewrite $z_A z_{A_i}^r$, yielding

$$(2) \quad S(\hat{e}_k^r, z_{[k]} z_A - z_{[k] \cap A} z_{[k] \cup A}) = z_A^{r+1} + \sum_j z_{A \cap A_j} z_{A_j}^{r-1} z_{A \cup A_j}$$

where the sum is over all j such that $|A_j| = k$, $A_j \neq A$, and $A \cap A_j \neq \emptyset$, since any terms involving z_\emptyset are elements of $J_{r,n}$. The observation that $|A| < |A \cup A_j|$ for all such j implies that z_A^{r+1} is the leading term of this polynomial, as desired.

Assume that we have added all prior S -polynomial calculations to the generators of $J_{r,n}$. To obtain terms of the form $z_A^r z_B$, where $A \subset B$ with $\max(A) < \min(B \setminus A)$ as listed in the fifth bullet of Proposition 4.2, let $|A| = k$. We compute the S -polynomial of \hat{e}_k^r and the generator of I_n with leading term $z_{[k]} z_B$. Note that $z_{[k]} z_B$ is the leading term of a generator of I_n , since by assumption $A \neq [k]$ thus if $[k] \subset B$ this would violate the condition $\max(A) < \min(B \setminus A)$. We compute:

$$\begin{aligned} & S(\hat{e}_k^r, z_{[k]} z_B - z_{[k] \cap B} z_{[k] \cup B}) \\ &= \frac{z_{[k]}^r z_B}{z_{[k]}^r} \left(z_{[k]}^r + z_{A_1}^r + z_{A_2}^r + \cdots + z_A^r + \cdots + z_{A_{\binom{n}{k}-1}}^r \right) \\ & \quad - \frac{z_{[k]}^r z_B}{z_{[k]}^r z_B} (z_{[k]} z_B - z_{[k] \cap B} z_{[k] \cup B}) \\ &= z_B \left(z_{A_1}^r + z_{A_2}^r + \cdots + z_A^r + \cdots + z_{A_{\binom{n}{k}-1}}^r \right) + z_{[k]}^{r-1} z_{[k] \cap B} z_{[k] \cup B} \end{aligned}$$

We have the ordering

$$z_{A_1}^r z_B > z_{A_2}^r z_B > \cdots > z_A^r z_B > \cdots > z_{A_{\binom{n}{k}-1}}^r z_B > z_{[k]}^{r-1} z_{[k] \cap B} z_{[k] \cup B}.$$

Moreover, by the condition $\max(A) < \min(B \setminus A)$ and the use of lexicographic order on subsets, we know that $A_i \not\subset B$ for all i such that $z_{A_i}^r z_B > z_A^r z_B$, which implies that $z_{A_i} z_B$ is a leading term of a polynomial in I_n . Applying $z_{A_i} z_B - z_{A_i \cap B} z_{A_i \cup B} \in J_{r,n}$ to the term $z_{A_i}^r z_B$ will produce $z_{A_i \cap B} z_{A_i}^{r-1} z_{A_i \cup B} < z_A z_B$. Therefore, we will have

$$\begin{aligned} & S(\hat{e}_k^r, z_{[k]} z_B - z_{[k] \cap B} z_{[k] \cup B}) = \\ & z_A^r z_B + \sum_j z_{A_j}^r z_B + \sum_m z_{A_m \cap B} z_{A_m}^{r-1} z_{A_m \cup B} \end{aligned}$$

where the first sum is over all j so that $|A_j| = |A|$, $A_j \neq A$, and $A_j \subset B$, which implies that $z_A > z_{A_j}$ by condition $\max(A) < \min(B \setminus A)$. The second sum is over all m such that $|A_m| = |A|$ where A_m and B are a Sperner 2-pair with $A_m \cap B \neq \emptyset$, as if the intersection was empty then the resulting term would be a multiple of z_\emptyset and hence an element of $J_{r,n}$. It follows from a simple cardinality argument that $z_{A_m \cup B} < z_B$, and thus $z_{A_m}^{r-1} z_{A_m \cup B}$ is a leading term in $J_{r,n}$.

Assume again that we have added all prior S -polynomial calculations to the generators of $J_{r,n}$. To obtain terms of the form $z_A z_B^r$ where there is an ℓ such that $[\ell] \not\subset A$, $[\ell] \subset B$ and $B \setminus A \subset [\ell]$, as listed in the sixth bullet of Proposition 4.2, let $|B| = k$. We compute the S -polynomial of \hat{e}_k^r and the generator of I_n with leading term $z_A z_{[k]}$, which is a leading term since there exists an element

$x \in [\ell] \subset [k]$ such that $x \notin A$ and there also exists $y = \max(A) = \max(B) \notin [k]$:

$$\begin{aligned}
& S(\hat{e}_k^r, z_A z_{[k]} - z_{A \cap [k]} z_{A \cup [k]}) \\
&= \frac{z_A z_{[k]}^r}{z_{[k]}^r} \left(z_{[k]}^r + z_{B_1}^r + \cdots + z_B^r + \cdots + z_{B_{\binom{n}{k}-1}}^r \right) \\
&\quad - \frac{z_A z_{[k]}^r}{z_A z_{[k]}^r} (z_A z_{[k]} - z_{A \cap [k]} z_{A \cup [k]}) \\
&= z_A \left(z_{B_1}^r + \cdots + z_B^r + \cdots + z_{B_{\binom{n}{k}-1}}^r \right) + z_{[k]}^{r-1} z_{A \cap [k]} z_{A \cup [k]}
\end{aligned}$$

which yields the term order of

$$z_A z_{B_1}^r > z_A z_{B_2}^r > \cdots > z_A z_B^r > \cdots > z_A z_{B_{\binom{n}{k}-1}}^r > z_{[k]}^{r-1} z_{A \cap [k]} z_{A \cup [k]}.$$

Note that $A \not\subset B_i$ for all i such that $z_{B_i} > z_B$, which implies that $z_A z_{B_i}$ is the leading term of a polynomial in I_n . As in our previous cases, this leads to the calculation

$$\begin{aligned}
& S(\hat{e}_k^r, z_A z_{[k]} - z_{A \cap [k]} z_{A \cup [k]}) = \\
& z_A z_B^r + \sum_j z_A z_{B_j}^r + \sum_m z_{A \cap B_m} z_{B_m}^{r-1} z_{A \cup B_m}
\end{aligned}$$

where the first sum is over all j such that $|B_j| = |B|$, $B \neq B_j$, and $A \subset B_j$. The second sum is over all m such that A and B_m are a Sperner 2-pair with $A \cap B_m \neq \emptyset$. Also, $|A \cup [k]| > |B|$. Ergo, we have $z_A z_B^r$ as the leading term.

Our final case is to obtain the terms listed in the seventh bullet of Proposition 4.2, i.e. those of type $z_{A_1} z_{A_2}^r z_{A_3}$ where $A_1 \subset A_2 \subset A_3$ and $\max(A_2 \setminus A_1) < \min(A_3 \setminus A_2)$ with $A_i \neq [j]$ for all j . Assume that we have added all prior S -polynomials to the generators of $J_{r,n}$. We consider the S -polynomial for the elements $z_{A_2} z_{A_1 \cup (A_3 \setminus A_2)} - z_{A_1} z_{A_3}$ and the generator from (2) given by $z_{A_2}^{r+1} + \sum_j z_{A_2 \cap C_j} z_{C_j}^{r-1} z_{A_2 \cup C_j}$ where $|C_j| = |A_2| = k$, $A_2 \neq C_j$, and $A_2 \cap C_j \neq \emptyset$. Let $B := A_1 \cup (A_3 \setminus A_2)$ for convenience of notation, and compute:

$$\begin{aligned}
& S \left(z_{A_2}^{r+1} + \sum_j z_{A_2 \cap C_j} z_{C_j}^{r-1} z_{A_2 \cup C_j}, z_{A_2} z_B - z_{A_1} z_{A_3} \right) \\
&= \frac{z_{A_2}^{r+1} z_B}{z_{A_2}^{r+1}} \left(z_{A_2}^{r+1} + \sum_j z_{A_2 \cap C_j} z_{C_j}^{r-1} z_{A_2 \cup C_j} \right) \\
&\quad - \frac{z_{A_2}^{r+1} z_B}{z_{A_2} z_B} (z_{A_2} z_B - z_{A_1} z_{A_3}) \\
&= z_B \sum_j z_{A_2 \cap C_j} z_{C_j}^{r-1} z_{A_2 \cup C_j} + z_{A_1} z_{A_2}^r z_{A_3} \\
&= z_{A_1 \cup (A_3 \setminus A_2)} \sum_j z_{A_2 \cap C_j} z_{C_j}^{r-1} z_{A_2 \cup C_j} + z_{A_1} z_{A_2}^r z_{A_3}
\end{aligned}$$

We now wish to show the $z_{A_1} z_{A_2}^r z_{A_3}$ is the leading term. Consider the terms involving C_j . There are three possible cases

1. $|A_2 \cup C_j| > |A_3|$
2. $|A_2 \cup C_j| < |A_3|$
3. $|A_2 \cup C_j| = |A_3|$

which we consider individually.

Case 1: If we have that $|A_2 \cup C_j| > |A_3|$, then we have $z_{A_1} z_{A_2}^r z_{A_3} >_{\text{grevlex}} z_{A_2 \cap C_j} z_{A_1 \cup (A_3 \setminus A_2)} z_{C_j}^{r-1} z_{A_2 \cup C_j}$ immediately by the definition of graded reverse lexicographic order.

Case 2: Suppose that we have $|A_2 \cup C_j| < |A_3|$. Note that this implies that there exists $x \in A_3$ such that $x \notin A_2 \cup C_j$ and hence $x \in A_1 \cup (A_3 \setminus A_2)$. We also have $y \in A_2 \cup C_j$ such that $y \notin A_1 \cup (A_3 \setminus A_2)$. Hence, we have that $A_1 \cup (A_3 \setminus A_2)$ and $A_2 \cup C_j$ are a Sperner 2-pair. This implies that we can replace the monomial $z_{A_2 \cap C_j} z_{C_j}^{r-1} z_{A_1 \cup (A_3 \setminus A_2)} z_{A_2 \cup C_j}$ with the monomial

$$\begin{aligned} & z_{A_2 \cap C_j} z_{C_j}^{r-1} z_{(A_1 \cup (A_3 \setminus A_2)) \cap (A_2 \cup C_j)} z_{(A_1 \cup (A_3 \setminus A_2)) \cup (A_2 \cup C_j)} \\ &= z_{A_2 \cap C_j} z_{C_j}^{r-1} z_{A_1 \cup (C_j \cap (A_3 \setminus A_2))} z_{A_3 \cup C_j} \end{aligned}$$

It is clear that $|A_3 \cup C_j| \geq |A_3|$. If the inequality is strict, then we are done. If $A_3 \cup C_j = A_3$, note that $C_j \subset A_3$ and that $C_j \cap (A_3 \setminus A_2) \neq \emptyset$ since $|C_j| = |A_2|$. We will now consider the variable $z_{A_1 \cup (C_j \cap (A_3 \setminus A_2))}$. We note that two subcases arise:

- 2.i.** $A_1 \cup (C_j \cap (A_3 \setminus A_2)) = A_1 \cup C_j$ (equivalently $C_j \cap A_1 = C_j \cap A_2$)
- 2.ii.** $A_1 \cup (C_j \cap (A_3 \setminus A_2))$ and $A_2 \cap C_j$ are a Sperner 2-pair.

Subcase 2.i: Note that $|A_1 \cup C_j| \geq |A_2|$ with equality occurring if $A_1 \cup C_j = C_j$. If the inequality is strict, we are done. If $A_1 \cup C_j = C_j$, then $|C_j| = |A_2|$, but since $C_j \cap A_1 = C_j \cap A_2$ and $C_j \cap (A_3 \setminus A_2) \neq \emptyset$, the condition $\max(A_2 \setminus A_1) < \min(A_3 \setminus A_2)$ implies that A_2 is lexicographically before C_j , which is desired.

Subcase 2.ii: The existence of such a Sperner 2-pair allows us to replace the monomial through division by

$$\begin{aligned} & z_{(A_1 \cup (C_j \cap (A_3 \setminus A_2))) \cap (A_2 \cap C_j)} z_{(A_1 \cup (C_j \cap (A_3 \setminus A_2))) \cup (A_2 \cap C_j)} z_{C_j}^{r-1} z_{A_3} \\ &= z_{(A_1 \cup (C_j \cap (A_3 \setminus A_2))) \cap (A_2 \cap C_j)} z_{A_1 \cup C_j} z_{C_j}^{r-1} z_{A_3} \end{aligned}$$

Showing the desired outcome is now identical to the argument in Subcase 2.i.

Case 3: Suppose that we have $|A_2 \cup C_j| = |A_3|$. In this case, it is sufficient to consider the following three plausible sub-cases.

- 3.i.** $A_2 \cup C_j$ and $A_1 \cup (A_3 \setminus A_2)$ are a Sperner 2-pair.
- 3.ii.** The subcase 3.i. is false, but $A_2 \cap C_j$ and $A_1 \cup (A_3 \setminus A_2)$ are a Sperner 2-pair.
- 3.iii.** $A_2 \cap C_j$, $A_2 \cup C_j$, and $A_1 \cup (A_3 \setminus A_2)$ have no Sperner 2-pairs between them.

Subcase 3.i: Suppose we have that the sets $A_2 \cup C_j$ and $A_1 \cup (A_3 \setminus A_2)$ are a Sperner 2-pair. This means that via division, we can replace the existing monomial $z_{A_2 \cap C_j} z_{C_j}^{r-1} z_{A_1 \cup (A_3 \setminus A_2)} z_{A_2 \cup C_j}$ with the monomial

$$\begin{aligned} & z_{A_2 \cap C_j} z_{C_j}^{r-1} z_{(A_2 \cup C_j) \cap (A_1 \cup (A_3 \setminus A_2))} z_{(A_2 \cup C_j) \cup (A_1 \cup (A_3 \setminus A_2))} \\ &= z_{A_2 \cap C_j} z_{C_j}^{r-1} z_{(A_2 \cup C_j) \cap (A_1 \cup (A_3 \setminus A_2))} z_{A_3 \cup C_j} \end{aligned}$$

By virtue of the Sperner 2-pair assumptions, we have that there exists $x \in C_j$ such that $x \notin A_3$, which yields $|A_3 \cup C_j| > |A_3|$ and hence

$$z_{A_1} z_{A_2}^r z_{A_3} >_{\text{grevlex}} z_{A_2 \cap C_j} z_{C_j}^{r-1} z_{(A_2 \cup C_j) \cap (A_1 \cup (A_3 \setminus A_2))} z_{A_3 \cup C_j}$$

and we are done.

Subcase 3.ii: Suppose that $A_2 \cap C_j$ and $A_1 \cup (A_3 \setminus A_2)$ are a Sperner 2-pair, but that $A_2 \cup C_j$ and $A_1 \cup (A_3 \setminus A_2)$ are not. Then note that we have $A_1 \cup (A_3 \setminus A_2) \subset A_2 \cup C_j$, which implies that $A_3 \setminus A_2 \subset C_j$, and hence $A_2 \cup C_j = A_3$ by the cardinality assumption. Now, by the existence of the Sperner 2-pair, we can replace via division the existing monomial $z_{A_2 \cap C_j} z_{C_j}^{r-1} z_{A_1 \cup (A_3 \setminus A_2)} z_{A_2 \cup C_j}$ with the monomial

$$z_{(A_2 \cap C_j) \cap (A_1 \cup (A_3 \setminus A_2))} z_{(A_2 \cap C_j) \cup (A_1 \cup (A_3 \setminus A_2))} z_{C_j}^{r-1} z_{A_3}$$

Moreover, notice that $C_j \subseteq ((A_2 \cap C_j) \cup (A_1 \cup (A_3 \setminus A_2)))$. If the equality is strict, we have that $|A_2| < |((A_2 \cap C_j) \cup (A_1 \cup (A_3 \setminus A_2)))|$ and we are done. If we have equality, then we know $|A_2| = |((A_2 \cap C_j) \cup (A_1 \cup (A_3 \setminus A_2)))|$. By the assumption that $\max(A_2 \setminus A_1) < \min(A_3 \setminus A_2)$, this implies that A_2 is lexicographically before $((A_2 \cap C_j) \cup (A_1 \cup (A_3 \setminus A_2)))$. Thus we will have

$$z_{A_1} z_{A_2}^r z_{A_3} >_{\text{grevlex}} z_{(A_2 \cap C_j) \cup (A_1 \cup (A_3 \setminus A_2))} z_{(A_2 \cap C_j) \cup (A_1 \cup (A_3 \setminus A_2))}^{r-1} z_{A_3}$$

which is as desired.

Subcase 3.iii: Suppose that the sets $A_2 \cap C_j$, $A_2 \cup C_j$, and $A_1 \cup (A_3 \setminus A_2)$ have no Sperner 2-pairs between them. This implies the following containment

$$A_2 \cap C_j \subset A_1 \cup (A_3 \setminus A_2) \subset A_2 \cup C_j = A_3$$

because $A_2 \cap C_j \subseteq A_1$ and $A_3 \subseteq A_2 \cup C_j$, which follows from the necessary containment and the fact that these sets have the same cardinality. These observations allow us to conclude that $C_j \subseteq A_1 \cup (A_3 \setminus A_2)$. If the containment is strict, we have that $|A_1 \cup (A_3 \setminus A_2)| > |A_2|$ and we are done. If equality holds, we have $|A_1 \cup (A_3 \setminus A_2)| = |A_2|$. However, the assumed condition that $\max(A_2 \setminus A_1) < \min(A_3 \setminus A_2)$ implies that A_2 is lexicographically before $A_1 \cup (A_3 \setminus A_2)$. Thus, we have that

$$z_{A_1} z_{A_2}^r z_{A_3} >_{\text{grevlex}} z_{A_2 \cap C_j} z_{A_1 \cup (A_3 \setminus A_2)} z_{C_j}^{r-1} z_{A_3}$$

which is our desired result.

Given all of the above, we can conclude that

$$S \left(z_{A_2}^r + \sum_j z_{A_2 \cap C_j} z_{C_j}^{r-1} z_{A_2 \cup C_j}, z_{A_2} z_B - z_{A_1} z_{A_3} \right) = z_{A_1} z_{A_2}^r z_{A_3} + p_{A_1 A_2^r A_3}$$

where $p_{A_1 A_2^r A_3}$ is a polynomial with $\text{LT}(p_{A_1 A_2^r A_3}) < z_{A_1} z_{A_2}^r z_{A_3}$.

We have now shown that all of our desired leading terms appear through the optimized Buchberger Algorithm. Because of our previous dimension calculation for $T_n/J_{r,n}$, we know that no additional leading terms can result from further computations, thus we have a Gröbner basis. \square

We have now shown Theorem 4.1, as it follows immediately from Proposition 4.2 and Theorem 4.4.

5. COMBINATORIAL IDENTITIES

We will now compute multigraded Hilbert series to prove Theorems 1.1 and 1.2. It is straightforward [5] to show that $\text{Hilb}(R_n; t, q) = \sum_{k \geq 1} [k+1]_q^n t^k$, which we assume for both of the following proofs. We will use the notation $\mathcal{C}_{r,n}$ introduced in the proof of Theorem 4.4.

Proof of Theorem 1.1. For an element $z_A \in R_n$, we define $\deg(z_A) = tq^{|A|}$. We have that

$$\text{Hilb}(R_n; t, q) = \frac{\text{Hilb}(T_n/J_{1,n}; t, q)}{\prod_{j=0}^n (1 - tq^j)}$$

which follows from the elementary exercise that

$$\text{Hilb}(\mathcal{C}_{1,n}; t, q) = \frac{1}{(1-t)(1-tq) \cdots (1-tq^n)}.$$

We now compute

$$\text{Hilb}(T_n/J_{1,n}; t, q) = \sum_{\pi \in S_n} \deg(\hat{a}_\pi) = \sum_{\pi \in S_n} t^{\text{des}(\pi)} q^{\text{maj}(\pi)}$$

which concludes the proof. \square

Proof of Theorem 1.2. Similarly to the previous argument, for any $z_A \in R_n$, we define $\deg(z_A) = tq^{|A|}$. We have that

$$\text{Hilb}(R_n; t, q) = \frac{\text{Hilb}(T_n/J_{r,n}; t, q)}{(1-t) \prod_{j=1}^n (1-t^r q^{rj})}$$

because as in the previous proof it is straightforward to show that

$$\text{Hilb}(\mathcal{C}_{r,n}; t, q) = \frac{1}{(1-t)(1-t^r q^r)(1-t^r q^{2r}) \cdots (1-t^r q^{rn})}.$$

Hence, we compute

$$\begin{aligned} \text{Hilb}(T_n/J_{r,n}; t, q) &= \sum_{(\pi, X) \in \mathbb{Z}_r \wr S_n} \deg(b_{(\pi, X)}^r) \\ &= \sum_{(\pi, X) \in \mathbb{Z}_r \wr S_n} t^{\text{des}(\pi)} q^{\text{maj}(\pi)} t^{|X|} q^{\sum_{i \in X} i} \\ &= \sum_{(\rho, \epsilon) \in \mathbb{Z}_r \wr S_n} t^{\text{ndes}(\rho, \epsilon)} q^{\text{nmajor}(\rho, \epsilon)}, \end{aligned}$$

completing the proof. \square

6. CONCLUDING REMARKS

It is worth mentioning that when $r = 1$ there is a graded S_n -module isomorphism between $T_n/J_{1,n}$ and $\mathbb{C}[x_1, x_2, \dots, x_n]/\mathcal{I}_n$.

Theorem 6.1. *The map $\varphi : T_n/J_{1,n} \rightarrow \mathbb{C}[x_1, x_2, \dots, x_n]/\mathcal{I}_n$ defined by algebraically extending $z_A + J_{1,n} \mapsto \prod_{i \in A} x_i + \mathcal{I}_n$ is an S_n -isomorphism.*

Proof. Consider $T_n/J_{1,n}$ under the q -grading used in the multigrading for Section 5, i.e. $\deg(z_A) = |A|$. Let $\mathbb{C}[x_1, x_2, \dots, x_n]/\mathcal{I}_n$ be graded by total degree. It is clear the φ respects grading, by definition. Moreover, it is clear that φ is an algebra isomorphism, since

$$\begin{aligned} \varphi(z_A + J_{1,n}) \cdot \varphi(z_B + J_{1,n}) &= (\prod_{i \in A} x_i + \mathcal{I}_n) \cdot (\prod_{j \in B} x_j + \mathcal{I}_n) \\ &= (\prod_{i \in A} x_i) \cdot (\prod_{j \in B} x_j) + \mathcal{I}_n \\ &= \varphi(z_A z_B + J_{1,n}) \end{aligned}$$

which implies $\varphi(\hat{a}_\pi + J_{1,n}) = a_\pi + \mathcal{I}_n$ for all $\pi \in S_n$.

Now to show that the action is preserved. Consider $z_A + J_{1,n}$ and $\sigma \in S_n$, and observe that

$$\begin{aligned} \sigma \circ \varphi(z_A + J_{1,n}) &= \sigma(\prod_{i \in A} x_i) + \mathcal{I}_n \\ &= \prod_{i \in A} x_{\sigma(i)} + \mathcal{I}_n \\ &= \prod_{i \in \sigma(A)} x_i + \mathcal{I}_n \\ &= \varphi(z_{\sigma(A)} + J_{1,n}) \\ &= \varphi \circ \sigma(z_A + J_{1,n}). \end{aligned}$$

\square

It would be interesting to determine if the representation-theoretic results of [2] are easier to establish in the context of $T_n/J_{1,n}$ rather than $\mathbb{C}[x_1, x_2, \dots, x_n]/\mathcal{I}_n$.

REFERENCES

- [1] Ron M. Adin, Francesco Brenti, and Yuval Roichman. Descent numbers and major indices for the hyperoctahedral group. *Adv. in Appl. Math.*, 27(2-3):210224, 2001. Special issue in honor of Dominique Foata's 65th birthday (Philadelphia, PA, 2000).
- [2] Ron M. Adin, Francesco Brenti, and Yuval Roichman. Descent representations and multivariate statistics. *Trans. Amer. Math. Soc.*, 357(8):3051-3082, 2004.
- [3] Eli Bagno. Euler-Mahonian parameters on colored permutation groups. *Sém. Lothar. Combin.*, 51:Art. B51f, 16pp. (electronic), 2004/05

- [4] Eli Bagno and Riccardo Biagioli. Colored-descent representations of complex reflection groups $G(r, p, n)$. *Israel J. Math.*, 160:317-347, 2007.
- [5] Matthias Beck and Benjamin Braun. Euler-Mahonian statistics via polyhedral geometry. *Advances in Mathematics*, 244(0):925-954, 2013.
- [6] Leonard Carlitz. A combinatorial property of q -Eulerian numbers. *Amer. Math. Monthly*, 82:51-54, 1975.
- [7] David Cox, John Little, and Donal O'Shea. *Ideals, Varieties, and Algorithms: An Introduction to Computational Algebraic Geometry and Commutative Algebra* 3rd ed., Springer-Verlag, New York (2007).
- [8] Viviana Ene. Syzygies of Hibi Rings. *Acta Math. Vietnam.* 40(3): 403-446, 2015
- [9] Leonhard Euler. Remarques sur un beau rapport entre les series des puissances tant direct que reciproques. *Memoires de l'academie des sciences de Berlin*, 17:83106, 1768.
- [10] A. M. Garsia and D. Stanton. Group actions of Stanley-Reisner rings and invariants of permutation groups. *Adv. in Math.*, 51(2):107-201, 1984.
- [11] Jürgen Herzog and Takayuki Hibi. *Monomial Ideals*. Graduate Texts in Mathematics, vol. **260**. Springer, Berlin (2010).
- [12] Takayuki Hibi. Distributive lattices, affine semigroup rings, and algebras with straightening laws. *Commutative algebra and combinatorics (Kyoto, 1985)*, 93-109, Advanced Studies in Pure Mathematics, 11, North-Holland, Amsterdam, 1987.
- [13] Takayuki Hibi. *Algebraic Combinatorics on Convex Polytopes*. Carlsaw, Glebe (1992).
- [14] M. Hochster. Rings of invariants of tori, CohenMacaulay rings generated by monomials, and polytopes. *Ann. of Math.* (2), 96:318337, 1972.
- [15] Jia Huang. 0-Hecke algebra action on the Stanley-Reisner ring of the Boolean algebra. *Ann. Comb.* 19(2):293-323, 2015.
- [16] Ezra Miller and Bernd Sturmfels. *Combinatorial Commutative Algebra*. Graduate Texts in Mathematics, vol. **227**. Springer, Berlin (2005).
- [17] Thomas W. Pensyl and Carla D. Savage. Lecture hall partitions and the wreath products $C_k \wr S_n$. *Integers*, 12B(Proceedings of the Integers Conference 2011):Paper No. A10, 18, 2012/13
- [18] John Shareshian and Michelle L. Wachs. Eulerian quasisymmetric functions. *Adv. Math.*, 225(6):2921-2966, 2010.
- [19] Richard Stanley. *Enumerative Combinatorics, Volume I* 2nd ed., Cambridge Studies in Advanced Mathematics, no. 49, Cambridge University Press, New York (2012).
- [20] Bernd Sturmfels. *Algorithms in Invariant Theory* 2nd ed., Texts and Monographs in Symbolic Computation, Springer-Wein, New York (2008).

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF KENTUCKY, LEXINGTON, KY 40506-0027
E-mail address: benjamin.braun@uky.edu

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF KENTUCKY, LEXINGTON, KY 40506-0027
E-mail address: mccabe.olsen@uky.edu